Chapter 13.

Complex numbers.

Consider the three quadratic equations

$$(x-3)(2x+1) = 0,$$
$$x^2 - 2x - 8 = 0,$$
and
$$x^2 + 6x + 3 = 0.$$

Asked to solve these equations you might opt to solve each using the ability of your calculator but hopefully you would realise that:

☞ the first, being in factorised form, is readily solved without the assistance of a calculator. (To give: $x = 3$, $x = -0{\cdot}5$)

☞ the second is readily factorized and hence can also be solved without the assistance of a calculator. (To give: $x = 4$, $x = -2$)

The third equation, not being in factorized form and not being readily factorized, could be solved using the ability of some calculators to solve such equations, as the display on the right suggests, or by completing the square (as shown below left), or by use of the quadratic formula (as shown below right).

solve($x^2+6{\cdot}x+3=0,x$)
$\{x=-5.449489743, x=-0.5505102572\}$
solve($x^2+6{\cdot}x+3=0,x$)
$\{x=-\sqrt{6}-3, x=\sqrt{6}-3\}$

Completing the square.

$$x^2 + 6x + 3 = 0$$

Create a "gap"

$$x^2 + 6x = -3$$

Insert the square of half the coeff of x

$$x^2 + 6x + 3^2 = -3 + 3^2$$
$$x^2 + 6x + 9 = 6$$
$$(x+3)^2 = 6$$
$$x + 3 = \pm\sqrt{6}$$
$$x = -3 \pm \sqrt{6}$$

Use of the formula.

Comparing $x^2 + 6x + 3 = 0$

with $ax^2 + bx + c = 0$

gives $a = 1$, $b = 6$ and $c = 3$.

Using $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

gives $x = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(3)}}{2(1)}$

$= \dfrac{-6 \pm \sqrt{24}}{2}$

$= \dfrac{-6 \pm 2\sqrt{6}}{2}$

$\therefore \quad x = -3 \pm \sqrt{6}$

Now consider $x^2 + 6x + 13 = 0.$

If we approach solving this quadratic equation by using the quadratic formula, we encounter the square root of a negative number and correctly conclude that there are "no real solutions":

$$\begin{aligned} \text{Comparing} \quad x^2 + 6x + 13 &= 0 \\ \text{with} \quad ax^2 + bx + c &= 0 \quad \text{gives } a = 1, b = 6 \text{ and } c = 13. \end{aligned}$$

$$\text{Thus} \qquad x = \frac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{-16}}{2}$$

$\therefore$ No real solutions.

However if we attempt to solve this same equation using a calculator we may be given a message indicating there are no solutions, as in the display below left, or we may be given solutions that involve " i ", as in the display below right. Which of these types of response we get depends on whether our calculator is set to solve for real solutions only or is set to include "complex" solutions.

solve($x^2+6\cdot x+13=0, x$)

No Solution

solve($x^2+6\cdot x+13=0, x$)

$\{x=-3-2\cdot \text{i},\ x=-3+2\cdot \text{i}\}$

Let us now consider what this "i" means in the "complex" solutions.

There are indeed no *real* solutions to the equation

$$x^2 + 6x + 13 = 0$$

but a calculator display showing complex solutions uses the concept of $\sqrt{-1}$ being an *imaginary*, non-real, number. Using i to represent $\sqrt{-1}$ gives us a way of representing the non-real solutions of an equation symbolically as shown on the next page.

With $a = 1, b = 6$ and $c = 13$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{-16}}{2}$$

$$= \frac{-6 \pm \sqrt{(16)(-1)}}{2}$$

$$= \frac{-6 \pm 4\sqrt{-1}}{2}$$

$$= \frac{-6 \pm 4\mathrm{i}}{2}$$

$$= -3 \pm 2\mathrm{i}$$

$$= -3 + 2\mathrm{i} \quad \text{or} \quad -3 - 2\mathrm{i}$$ as displayed earlier.

As you would be familiar with from your study of *Mathematics Methods*, the quantity $(b^2 - 4ac)$ is called the **discriminant** of the quadratic equation. Its value allows us to *discriminate* between the various types of solution the equation may have:

If $b^2 - 4ac > 0$ we have two real solutions.
If $b^2 - 4ac = 0$ there is just one solution, and it will be real.
If $b^2 - 4ac < 0$ we have two *complex* solutions.

Note: The solutions of an equation $f(x) = 0$ are also referred to as the *roots* of the equation.
The "just one solution" situation is sometimes referred to as a "repeated root" because the "two" solutions to the equation are the same, i.e. "repeated".

Complex numbers.

Numbers like $-3 + 2\mathrm{i}$ and $-3 - 2\mathrm{i}$ have the general form $a + b\mathrm{i}$ (sometimes written $x + \mathrm{i}y$) where a and b are real and $\mathrm{i} = \sqrt{-1}$. Such numbers are called **complex** numbers. They consist of a real part, a, and an imaginary part, b.

If	$z = a + b\mathrm{i}$	we say that the real part of z is a:	$\mathrm{Re}(z) = a$
		and the imaginary part of z is b:	$\mathrm{Im}(z) = b$.
Thus if	$z = 4 + 5\mathrm{i}$	then	$\mathrm{Re}(z) = 4$
		and	$\mathrm{Im}(z) = 5$.

Introducing an i to represent $\sqrt{-1}$ allows us to give more informative solutions to quadratics in which $(b^2 - 4ac)$ is negative, than simply saying "no real solutions". However, if that was the only benefit gained from expanding our number system to include the concept of a complex number it would hardly be worth the effort. We do make the effort though because, as you will find if you continue your mathematical studies, complex numbers do prove to be very useful in some branches of mathematics.

The idea of expanding our number system to include the concept of a complex number should not be seen as anything particularly strange. When you first started counting, the number system as far as you were concerned would have consisted only of the counting numbers:

$$1, \; 2, \; 3, \; 4, \; 5, \; \ldots$$

As the concepts in which you used number became more involved your number system needed to develop to include ways of representing fractions, zero and negatives. Irrational numbers like $\sqrt{2}$ and π then became necessary and now we need to expand the number system beyond $\mathbb{R}$ so that when we are asked to solve equations like

$$\begin{aligned} x^2 &= -4, \\ x^2 + 8 &= 0, \\ x^2 + 6x + 13 &= 0, \qquad \text{etc.} \end{aligned}$$

we can be more informative than simply saying *no real solutions.*

Example 1

Use the fact that if $ax^2 + bx + c = 0$

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

to determine the exact solutions of the following quadratic equations giving your answers in the form $d + e\text{i}$ where d and e are real numbers and $\text{i} = \sqrt{-1}$.

(a) $x^2 + 4x + 5 = 0$ (b) $2x^2 - 3x + 2 = 0$

(a) Comparing $x^2 + 4x + 5 = 0$

with $ax^2 + bx + c = 0$

gives $a = 1$, $b = 4$ and $c = 5$.

$$\begin{aligned} \text{Thus } x &= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{-4}}{2} \\ &= \frac{-4 \pm \sqrt{(4)(-1)}}{2} \\ &= \frac{-4 \pm 2\text{i}}{2} \\ &= -2 + \text{i} \text{ or } -2 - \text{i} \end{aligned}$$

(b) Comparing $2x^2 - 3x + 2 = 0$

with $ax^2 + bx + c = 0$

gives $a = 2$, $b = -3$ and $c = 2$.

$$\begin{aligned} \text{Thus } x &= \frac{3 \pm \sqrt{(-3)^2 - 4(2)(2)}}{2(2)} \\ &= \frac{3 \pm \sqrt{-7}}{4} \\ &= \frac{3 \pm \sqrt{(7)(-1)}}{4} \\ &= \frac{3 \pm \sqrt{7}\,\text{i}}{4} \\ &= \frac{3}{4} + \frac{\sqrt{7}}{4}\text{i} \text{ or } \frac{3}{4} - \frac{\sqrt{7}}{4}\text{i} \end{aligned}$$

Example 2

For the complex number $z = 2 + 3\text{i}$ state (a) Re(z),
and (b) Im(z).

If $z = 2 + 3\text{i}$ then (a) $\text{Re}(z) = 2$,
and (b) $\text{Im}(z) = 3$.

Note • Re(z) and Im(z) are both real numbers. Im(z) = 3, not 3i.

re(2+3·i)	
	2
im(2+3·i)	
	3

Exercise 13A

Write each of the following in the form ai where a is real and $\text{i} = \sqrt{-1}$.

1. $\sqrt{-25}$
2. $\sqrt{-144}$
3. $\sqrt{-9}$
4. $\sqrt{-49}$
5. $\sqrt{-400}$
6. $\sqrt{-5}$
7. $\sqrt{-8}$
8. $\sqrt{-45}$
9. For the complex number $z = 3 + 5\text{i}$ state (a) Re(z), (b) Im(z).
10. For the complex number $z = -2 + 7\text{i}$ state (a) Re(z), (b) Im(z).
11. For the complex number $z = 3 - \text{i}$ state (a) Re(z), (b) Im(z).

Use the fact that if $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to determine the exact solutions of the following quadratic equations giving your answers in the form $d + e\text{i}$ where d and e are real numbers and $\text{i} = \sqrt{-1}$.

12. $x^2 + 2x + 5 = 0$
13. $x^2 + 2x + 3 = 0$
14. $x^2 + 4x + 6 = 0$
15. $x^2 + 2x + 10 = 0$
16. $x^2 - 4x + 6 = 0$
17. $2x^2 - x + 1 = 0$
18. $2x^2 + x + 1 = 0$
19. $2x^2 + 6x + 5 = 0$
20. $2x^2 - 2x + 25 = 0$
21. $5x^2 - 2x + 13 = 0$
22. $x^2 - x + 1 = 0$
23. $5x^2 - 3x + 1 = 0$

Complex number arithmetic.

The following example demonstrates complex number arithmetic. Note that in each case the answer is given in the form $a + b\text{i}$ and especially note the following technique used to achieve this when one complex number is divided by another:

If the denominator is $(a + b\text{i})$, we multiply by $\dfrac{(a - b\text{i})}{(a - b\text{i})}$.

(I.e. we multiply by 1, but the 1 is written in a very specific and helpful form.)

Example 3

If $w = 2 + 3\text{i}$ and $z = 5 - 4\text{i}$ determine: (a) $w + z$ (b) $w - z$ (c) $3w - 2z$ (d) wz (e) z^2 (f) $\dfrac{w}{z}$

(a) $$\begin{aligned} w + z &= (2 + 3\text{i}) + (5 - 4\text{i}) \\ &= 7 - \text{i} \end{aligned}$$

(b) $$\begin{aligned} w - z &= (2 + 3\text{i}) - (5 - 4\text{i}) \\ &= 2 + 3\text{i} - 5 + 4\text{i} \\ &= -3 + 7\text{i} \end{aligned}$$

(c) $$\begin{aligned} 3w - 2z &= 3(2 + 3\text{i}) - 2(5 - 4\text{i}) \\ &= 6 + 9\text{i} - 10 + 8\text{i} \\ &= -4 + 17\text{i} \end{aligned}$$

(d) $$\begin{aligned} wz &= (2 + 3\text{i})(5 - 4\text{i}) \\ &= 10 - 8\text{i} + 15\text{i} - 12\text{i}^2 \\ &= 10 - 8\text{i} + 15\text{i} + 12 \\ &= 22 + 7\text{i} \end{aligned}$$

(e) $$\begin{aligned} z^2 &= (5 - 4\text{i})(5 - 4\text{i}) \\ &= 25 - 20\text{i} - 20\text{i} + 16\text{i}^2 \\ &= 25 - 20\text{i} - 20\text{i} - 16 \\ &= 9 - 40\text{i} \end{aligned}$$

(f) $$\begin{aligned} \frac{w}{z} &= \frac{(2 + 3\text{i})}{(5 - 4\text{i})} \\ \therefore \quad \frac{w}{z} &= \frac{(2 + 3\text{i})}{(5 - 4\text{i})} \frac{(5 + 4\text{i})}{(5 + 4\text{i})} \\ &= \frac{10 + 8\text{i} + 15\text{i} + 12\text{i}^2}{25 + 20\text{i} - 20\text{i} - 16\text{i}^2} \\ &= \frac{-2 + 23\text{i}}{41} \\ &= -\frac{2}{41} + \frac{23}{41}\text{i} \end{aligned}$$

Alternatively, and as the reader should verify, these same answers can be obtained from a calculator.

The conjugate of a complex number.
If $z = a + bi$ we say that $a - bi$ is the **conjugate** of z.

We use the symbol $\bar{z}$ for the conjugate of z.

Thus if $z = 2 + 3i$ then $\bar{z} = 2 - 3i$,
if $z = 5 - 7i$ then $\bar{z} = 5 + 7i$,
if $z = -2 + 8i$ then $\bar{z} = -2 - 8i$,
if $z = -3 - 4i$ then $\bar{z} = -3 + 4i$, etc.

Note that for any complex number z ($= a + bi$), both the sum $z + \bar{z}$
and the product $z\bar{z}$ are real.

Proof: If $z = a + bi$ then $\bar{z} = a - bi$. Hence

$$\begin{aligned} z + \bar{z} &= (a + bi) + (a - bi) \\ &= 2a \qquad \text{a real number.} \end{aligned}$$

and

$$\begin{aligned} z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 - abi + abi - b^2i^2 \\ &= a^2 + b^2 \qquad \text{a real number.} \end{aligned}$$

- The fact that the product $z\bar{z}$ is real is used when dividing complex numbers, as in example 3 part (f) shown earlier and in part (d) of the next example.

Example 4

If $z = 12 - 5i$ determine (a) $\bar{z}$ (b) $z\bar{z}$ (c) $z + \bar{z}$ (d) $\dfrac{z}{\bar{z}}$

(a) $\bar{z} = 12 + 5i$

(b)
$$\begin{aligned} z\bar{z} &= (12 - 5i)(12 + 5i) \\ &= 144 + 60i - 60i - 25i^2 \\ &= 169 \end{aligned}$$

(c)
$$\begin{aligned} z + \bar{z} &= (12 - 5i) + (12 + 5i) \\ &= 24 \end{aligned}$$

(d)
$$\begin{aligned} \frac{z}{\bar{z}} &= \frac{(12 - 5i)}{(12 + 5i)} \\ &= \frac{(12 - 5i)}{(12 + 5i)}\frac{(12 - 5i)}{(12 - 5i)} \\ &= \frac{119}{169} - \frac{120}{169}i \end{aligned}$$

Equal complex numbers.

When we say that two complex numbers, w and z, are equal then we mean that

$\text{Re}(w) = \text{Re}(z)$ **and** $\text{Im}(w) = \text{Im}(z)$.

Thus if $a - 3i = 5 + bi$ then $a = 5$
and $b = -3$.

Linear factors of quadratic polynomials.

A linear factor is of the form $ax + b$, i.e. a linear expression. Hence writing $x^2 + 6x - 16$ in the form $(x + 8)(x - 2)$ is expressing $x^2 + 6x - 16$ in terms of linear factors.

Now that we have been introduced to complex numbers these linear factors could involve complex numbers, as the next example shows.

Example 5

Express $x^2 + 2x + 5$ as the product of two linear factors.

Using the quadratic formula.

If $\quad x^2 + 2x + 5 = 0$

Then $\quad x = \dfrac{-2 \pm \sqrt{-16}}{2}$

$= -1 \pm 2i$

Thus $x^2 + 2x + 5 = [x - (-1 + 2i)]\ [x - (-1 - 2i)]$

$= (x + 1 - 2i)(x + 1 + 2i)$

Using completing the square.

$$\begin{aligned} x^2 + 2x + 5 &= x^2 + 2x \quad + 5 \\ &= x^2 + 2x + 1 + 5 - 1 \\ &= (x + 1)^2 + 4 \end{aligned}$$

(Write as difference of 2 squares)

$$\begin{aligned} &= (x + 1)^2 - (2i)^2 \\ &= (x + 1 - 2i)(x + 1 + 2i) \end{aligned}$$

Can your calculator factorise $x^2 + 2x + 5$ in this way?

Exercise 13B

Simplify

1. $(2 + 3i) + (5 - i)$
2. $(5 - 6i) - (2 + 4i)$
3. $(2 + 3i) - (5 - i)$
4. $(5 - 6i) + (2 + 4i)$
5. $2 + 3i - 5 - i$
6. $5 - 6i + 2 + 4i$
7. $(3 + i) + (4 - 2i) + (6 + 5i)$
8. $2(3 + 2i) + 3(2 + i)$
9. $5(2 + i) + 3(1 - i)$
10. $5(2 + i) - 3(1 - i)$
11. $3(1 - 5i) + 7i$
12. $3(1 - 5i) + 7$
13. $\text{Re}(2 + 3i) + \text{Re}(5 - 2i)$
14. $\text{Im}(-1 + 4i) + \text{Im}(3 + i)$
15. $(3 + 2i)(2 + 5i)$
16. $(1 + 3i)(3 + 2i)$
17. $(2 + i)(1 - i)$
18. $(-2 + 3i)(5 + i)$

Express each of the following in the form a + bi where a and b are real numbers.

19. $\frac{(3+2i)}{(1+5i)}$

20. $\frac{(3+i)}{(1-2i)}$

21. $\frac{4}{(1+3i)}$

22. $\frac{2i}{(1+4i)}$

23. $\frac{(-3+2i)}{(2+3i)}$

24. $\frac{(5+i)}{(2i+3)}$

25. If $w = 5 - 2i$ and $z = 4 + 3i$ determine exactly

(a) $w + z$ (b) $w - z$ (c) $3w - 2z$ (d) wz (e) z^2 (f) $\frac{w}{z}$

26. If $Z_1 = 3 + 5i$ and $Z_2 = 1 - 5i$ determine exactly

(a) $Z_1 + Z_2$ (b) $Z_2 - Z_1$ (c) $Z_1 + 3Z_2$ (d) $Z_1 Z_2$ (e) Z_1^2 (f) $\frac{Z_1}{Z_2}$

27. If $z = 24 - 7i$ determine

(a) $\bar{z}$ (b) $z + \bar{z}$ (c) $z\bar{z}$ (d) $\frac{z}{\bar{z}}$

28. If $z = 4 + 9i$ determine

(a) $\bar{z}$ (b) $z - \bar{z}$ (c) $2z + 3\bar{z}$ (d) $2z - 3\bar{z}$ (e) $z\bar{z}$ (f) $\frac{z}{\bar{z}}$

29. Given that $z = 2 + ci$, $w = d + 3i$, c and d are real numbers and $z = w$ determine c and d.

30. If $a + bi = (2 - 3i)^2$, where a and b are real numbers, determine a and b.

31. If $z = 5 - (c + 3)i$ (c real), $w = d + 1 + 7i$ (d real) and $z = w$ determine c and d.

32. If $(a + 3i)(5 - i) = p$ where a and p are real numbers, determine a and p.

33. State whether each of the following are correct for all complex z and w? (If your answer is "no" then give an example to support your claim.)

(a) If $w = \bar{z}$, the conjugate of z, then $\text{Im}(w) = -\text{Im}(z)$.

(b) If $\text{Im}(z) = -\text{Im}(w)$ then $w = \bar{z}$.

34. Express each of the following as the product of two linear factors. (Not all will involve complex numbers.)

(a) $x^2 - 4x + 13$ (b) $x^2 - 2x + 10$

(c) $x^2 - 6x + 1$ (d) $x^2 + 10x + 26$

(e) $x^2 + 14x + 53$ (f) $x^2 + 4x - 3$

35. (a) Use the quadratic formula to prove that if a quadratic equation has any non-real roots then it must have 2 and they must be conjugates of each other.
    (b) One root of $x^2 + bx + c = 0$ is $x = 3 + 2\text{i}$. Find b and c.
    (c) One root of $x^2 + dx + e = 0$ is $x = 5 - 3\text{i}$. Find d and e.

36. Simplify (a) $\dfrac{c + d\text{i}}{-c - d\text{i}}$ (b) $\dfrac{c + d\text{i}}{d - c\text{i}}$ (c) $\dfrac{c - d\text{i}}{-d - c\text{i}}$

37. Find all possible real number pairs p, q such that $\dfrac{3 + 5\text{i}}{1 + p\text{i}} = q + 4\text{i}$.

38. The complex numbers z and w are such that for the real variable x

$$(x - z)(x - w) = ax^2 + bx + c \quad \text{for real } a, b \text{ and } c.$$

    (a) Determine the value of a.
    (b) Prove that $(z + w)$ and (zw) must both be real.
    (c) By letting $z = p + q\text{i}$ and $w = r + s\text{i}$, prove that z and w must be the conjugates of each other.

39. Given that $z = \text{a} + \text{bi}$, $w = \text{c} + \text{di}$, $\bar{z}$ is the conjugate of z and $\bar{w}$ is the conjugate of w prove that each of the following are true.

    (a) $\bar{z}\,\bar{w} = \overline{zw}$ (b) $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\bar{z}}{\bar{w}}$

40. The complex number $a + b\text{i}$ can be expressed as the "ordered pair" (a, b).
Express each of the following in this form.
(a) $2 + 3\text{i}$ (b) $-5 + 6\text{i}$ (c) 7i (d) 3
In the following each ordered pair represents a complex number.
Simplify the following sums giving answers in ordered pair form.
(e) $(3, 8) + (-2, 1)$ (f) $(3, -5) + (3, 5)$
Simplify the following differences giving answers in ordered pair form.
(g) $(5, 3) - (2, 0)$ (h) $(2, 7) - (2, -7)$
Simplify the following products giving answers in ordered pair form.
(i) $(0, 2) \times (3, 5)$ (j) $(-3, 1) \times (-3, -1)$
Simplify the following quotients giving answers in ordered pair form.
(k) $(3, 0) \div (2, -4)$ (l) $(3, -8) \div (3, 8)$

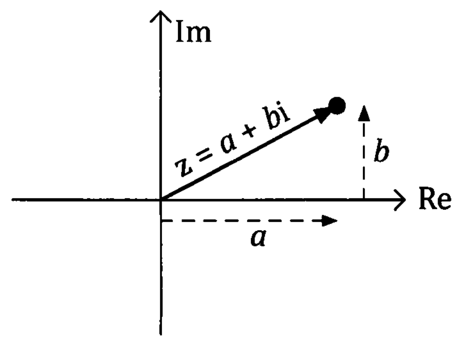
41. Showing full algebraic reasoning determine exactly, and in the form a + bi, the complex number z for which:

$$\frac{1}{z} = \frac{2 + 7\text{i}}{1 - \text{i}}$$

Argand Diagrams.

In this chapter we had written complex numbers in the form $a + b\text{i}$ until question 40 of the previous exercise, when we saw that they could also be written as an ordered pair (a, b). Clearly this ordered pair form of writing the complex number $a + b\text{i}$ is similar to the way we label a point on a graph by stating its coordinates. If instead of x and y axes we have real and imaginary axes we can use this similarity to provide us with a diagrammatic way of representing a complex number.

- ☛ Such a graphical representation is called an **Argand diagram**.
- ☛ The plane containing the real and imaginary axes is referred to as the complex plane.
- ☛ The complex number $a + b\text{i}$ can be thought of as the point (a, b) on the Argand diagram or as the vector from the origin to the point (a, b).



The diagram on the right uses these ideas to show the complex numbers

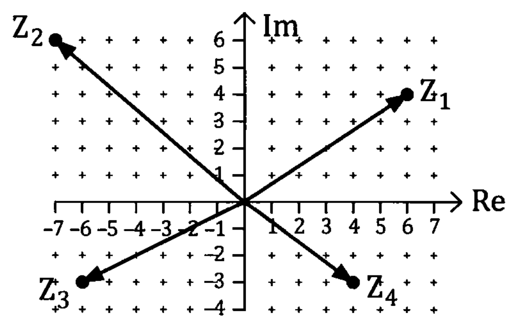
$$Z_1 = 6 + 4\text{i},$$

$$Z_2 = -7 + 6\text{i},$$

$$Z_3 = -6 - 3\text{i},$$

and

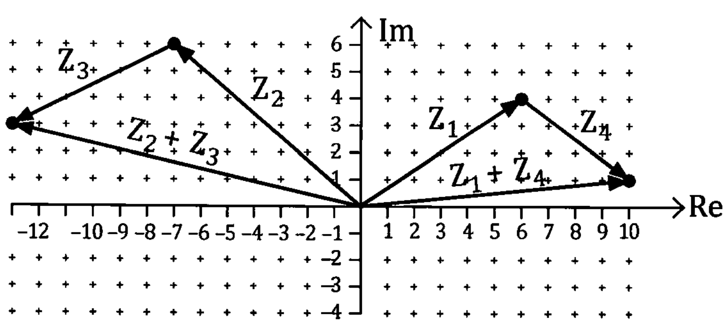
$$Z_4 = 4 - 3\text{i}.$$



Notice that we can then add complex numbers in the complex plane using the "nose to tail" method of vector addition.

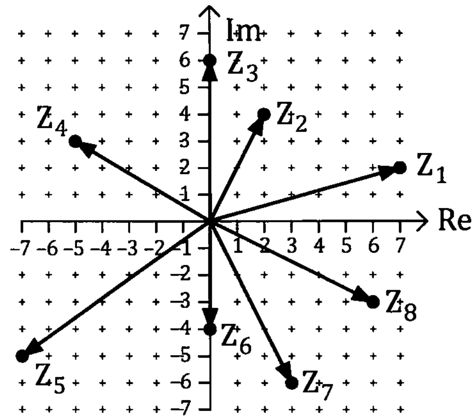
$$Z_2 + Z_3 = (-7 + 6\text{i}) + (-6 - 3\text{i}) = -13 + 3\text{i}$$

$$Z_1 + Z_4 = (6 + 4\text{i}) + (4 - 3\text{i}) = 10 + \text{i}$$

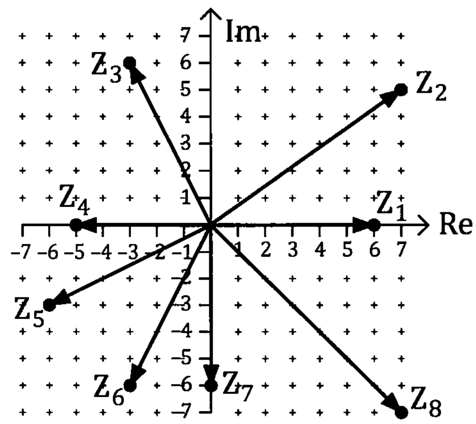


Exercise 13C

1. Express each of the complex numbers Z_1 to Z_8, shown on the Argand diagram on the right, in the form $a + b$i.



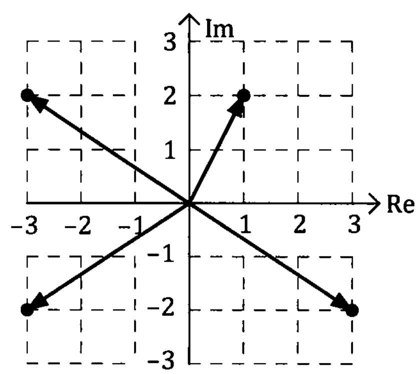
2. Express each of the complex numbers Z_1 to Z_8, shown on the Argand diagram on the right, in the ordered pair form (a, b).



3. Show the following numbers as vectors on a single Argand diagram.

$Z_1 = 4 + 3i$	$Z_2 = 4i$	$Z_3 = -3 + 5i$	$Z_4 = -5 + 2i$
$Z_5 = -3i$	$Z_6 = 2 - 4i$	$Z_7 = \overline{Z_1}$	$Z_8 = \overline{Z_3}$

4. The Argand diagram on the right shows four complex numbers Z_1, Z_2, Z_3 and Z_4.



Given that
- $\dfrac{\text{Re}(Z_1)}{\text{Im}(Z_1)} > 0$
- $\dfrac{\text{Re}(Z_2)}{\text{Im}(Z_2)} > 1$

and
- $Z_3 = \overline{Z_2}$

determine Z_1, Z_2, Z_3 and Z_4.

5. With $z = 3 + 5i$ display z, iz, $i^2 z$ and $i^3 z$ as vectors on a single Argand diagram.

6. Display Z_1, Z_2, ... Z_8 as vectors on a single Argand diagram where:

$Z_1 = 2 + i$	$Z_2 = (2 + i)(1 + i)$	$Z_3 = (2 + i)(1 + i)^2$
$Z_4 = (2 + i)(1 + i)^3$	$Z_5 = (2 + i)(1 + i)^4$	$Z_6 = (2 + i)(1 + i)^5$
$Z_7 = (2 + i)(1 + i)^6$	$Z_8 = (2 + i)(1 + i)^7$	

Miscellaneous Exercise Thirteen.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters in this unit, and the ideas mentioned in the preliminary work section at the beginning of the unit.

1. Simplify each of the following.

   (a) $(2 + 5i)(2 - 5i)$ (b) $(3 + i)(3 - i)$ (c) $(6 + 2i)(6 - 2i)$

   (d) $(3 + 4i)^2$ (e) $\dfrac{2 - 3i}{3 + i}$ (f) $\dfrac{3 + i}{2 - 3i}$

2. Given that $z = 2 - 3i$ and $w = -3 + 5i$ determine

   (a) $z + w$ (b) zw (c) $\bar{z}$, the conjugate of z

   (d) $\overline{zw}$ (e) z^2 (f) $(zw)^2$

   (g) the complex number p such that $\text{Re}(p) = \text{Re}\,\bar{z}$ and $\text{Im}(p) = \text{Im}\,\bar{w}$.

3. (a) $2x^3 - 5x^2 + 8x - 3 = (px - q)(x^2 + rx + 3)$ for real integers p, q and r. Determine p, q and r.

   (b) Without the assistance of a calculator, but using the fact that the quadratic equation $ax^2 + bx + c = 0$ has solutions given by

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   Find *exactly*, all values of x, real and complex, for which:

   $$2x^3 - 5x^2 + 8x - 3 = 0$$

4. Find (a) $(\text{Re}(2 + 3i))(\text{Re}(5 - 4i))$ (b) $\text{Re}((2 + 3i)(5 - 4i))$

5. Given that $z = 5\sqrt{2}\,i$ determine each of the following exactly.

   (a) $\bar{z}$ (b) z^2 (c) $(1 + z)^2$

6. Find the real numbers a and b given that $(a + bi)^2 = 5 - 12i$.

7. The triangle ABC is transformed to A'B'C' by the transformation matrix T where

   $$T = \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix}.$$

   (a) If A', B' and C' have coordinates (−1, 2), (10, −2) and (−4, −4) respectively, find the coordinates of A, B and C.

   (b) Draw triangles ABC and A'B'C' on grid paper and confirm that

   $$\text{Area } \Delta A'B'C' = |\det T|\ \text{Area } \Delta ABC.$$

8. The transformation matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ maps the point (1, −1) to (4, 1) and maps the point (2, −3) to (9, 1). Determine a, b, c and d.

9. Find all real or complex solutions to the equation $(z - 2 + 7i)^2 = -25$

10. A particular 2×2 matrix transforms all points in the x-y plane to a straight line.
    (a) What does this suggest about the determinant of the matrix?
    (b) Why must the straight line pass through the origin?

11. Given that $z = a + bi$ determine a and b given that $z + 2\bar{z} = 9 + 5i$, where $\bar{z}$ is the complex conjugate of z.

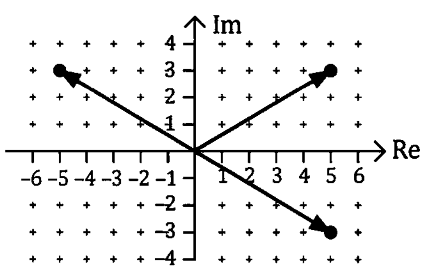
12. Using your calculator if you wish (a) express $(2 + 3i)^4$ in the form $a + bi$,
    (b) determine $\text{Im}((1 - 3i)^5)$.

13. Determine the quadratic equation $x^2 + bx + c = 0$ for which one of the solutions is $x = 2 + 3i$.

14. Determine the complex number z given that $3z + 2\bar{z} = 5 + 5i$, where $\bar{z}$ is the complex conjugate of z.

15. Showing full algebraic reasoning and giving your answer in the form
    $a + bi$,
    determine *exactly* the complex number z for which $z(2 - 3i) = 5 + i$.

16. The diagram on the right shows three complex numbers, w, $\overline{w}$ and z, with $\overline{w}$ representing the complex conjugate of w. Write z in the form $a + bi$ and determine z^2.



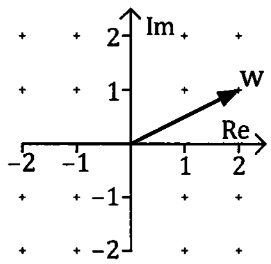
17. Prove that $\dfrac{\sin\theta}{\cos\left(\frac{1}{2}\theta\right)} = 2\sin\left(\frac{1}{2}\theta\right)$

18. Solve $\tan 2x + \tan x = 0$ for $0 \le x \le 360°$.

19. Prove that matrices of the form $\begin{bmatrix} a & b \\ ka & kb \end{bmatrix}$ transform **all** points to the straight line $y = kx$.

20. (a) Premultiply $\begin{bmatrix} -3 & 5 \end{bmatrix}$ by $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. (b) Postmultiply $\begin{bmatrix} -3 & 5 \end{bmatrix}$ by $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

21. With the complex number w as defined by the Argand diagram on the right, state which of the diagrams below show a complex number z for which:

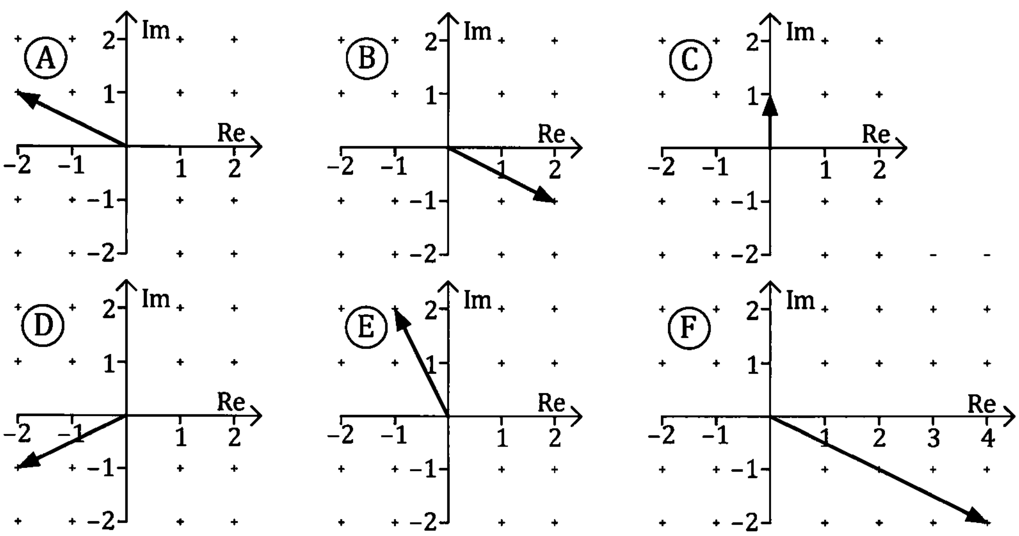


(a) $z = \overline{w}$ (b) $z + w$ is real

(c) zw is real (d) $\text{Im}(w) = \text{Im}(z)$

(e) $\text{Im}(w) = |\text{Im}(z)|$ (f) $|\text{Im}(w)| = \text{Im}(z)$

(g) $z = \text{i}w$ (h) $\dfrac{\overline{w}}{z}$ is real



22. Matrices A, B and C are all 2 × 2 matrices, matrix B is not a singular matrix and

$$A = BCB^{-1}.$$

Determine simplified expressions for (a) A^2 (b) A^3 (c) A^n.

23. Use the method of proof by induction to prove that

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + ... + n(n+1)(n+2) = \frac{n}{4}(n+1)(n+2)(n+3)$$

for all integer n, $n \geq 1$.

24. Prove, by induction, that $2^{n-1} + 3^{2n+1}$ is a multiple of seven for all integer n, $n \geq 1$.

25. Prove that $5^n + 3 \times 9^n$ is a multiple of 4 for all $n \geq 0$.

26. Prove that: $\sin\theta\,(\sin\theta + \sin 2\theta) = 1 + 2\cos\theta - \cos^2\theta - 2\cos^3\theta$.

27. Find all solutions to the equation

$$4\sin x\cos^2 x - \cos x = 0$$

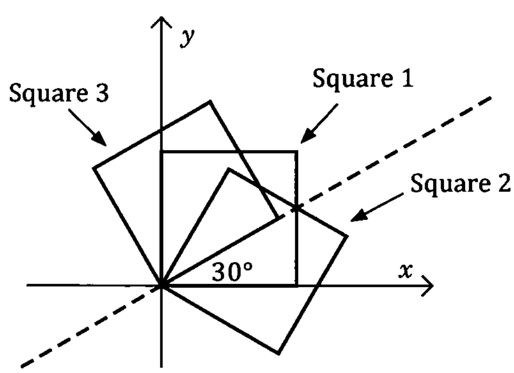
giving answers as exact values.

28. Write down
(a) the 2 × 2 transformation matrix that will perform a 30° anticlockwise rotation about the origin,
(b) the 2 × 2 transformation matrix that will perform a 60° anticlockwise rotation about the origin,
(c) the 2 × 2 transformation matrix that will reflect a shape in the line $y = \frac{\sqrt{3}}{3}x$.

(d) The diagram on the right shows square 1 reflected in the line

$$y = \frac{\sqrt{3}}{3}x$$

to give square 2, which is then rotated 60° anticlockwise about the origin to give square 3.
It would appear that square 3 could be obtained directly from square 1 by an anticlockwise rotation of 30° about the origin.



Use your answers from the earlier parts of this question to show that a reflection in the line $y = \frac{\sqrt{3}}{3}x$ followed by an anticlockwise rotation of 60° about the origin is not equivalent to an anticlockwise rotation of 30° about the origin.

Explain the apparent equivalence suggested by the diagram.